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A MODERN ZENO.

ZENO of Elea is famous mostly for his so-called "argument" to the effect that in a race between Achilles and a tortoise with the latter starting in advance a certain distance, Achilles can *never* overtake the tortoise although he may run many times as fast as his slow competitor. For, says Zeno, when Achilles reaches the spot where the tortoise started the other will have advanced to another point, and when Achilles has reached this second point the tortoise will have gone on to a third point and just so on and on the race will continue *ad infinitum,* the tortoise being always a little ahead. It is curious to see how this little *non sequitur* has perplexed people, many of them of excellent intellectual standing. Thus the famous logician Sir William Hamilton said the "argument" was unanswerable.

It is altogether beside my present theme to state wherein the catch lies, so I will merely say that the conclusion is no consequence at all from the premises. The "argument" stripped of its disguises is just this. Achilles can never overtake the tortoise because he cannot overtake it in any less time than it takes to do so.

Among men there is no habit more inveterate than the persuasion of each individual that he personally is immune from slips in reasoning. All around him during almost every day of his life he takes notice how badly other people reason without ever saying to himself that probably he is like other people in the same regard. So in view of the incontestable fact that men, yea, even men most eminent in intellectual power and cultivation, do sometimes err in reasoning, I make bold to confess a growing measure of misgiving as to certain geometrical results that have played and are still playing a conspicuous rôle in the mathematics of the present epoch. I refer to the so-called non-Euclidean geometry, and I propose to utter a little note of protest or rather of question. That is to say, some considerable study of the famous brochure of Lobatchevsky on parallels leaves my mind in such a state that I desire greatly some further instruction.

It is generally recognized that the problems of parallelism and of the angle-sum of the triangle are only two different aspects of a single problem. The solution of either involves the solution of the other. Lobatchevsky approaches the problem from a definition of parallelism. He adopts most of the fundamental definitions and conceptions of ordinary geometry and quite a number of the initial theorems. He defines the straight line in an original way saying: "A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place if it goes through two unmoving points in the surface (i. e., if we turn the surface containing it about two points of the line, the line does not move)."

Now it is one thing to give us an idea of an object and quite another to so define it that its essential quality or qualities shall be definitely stated. Lobatchevsky's definition is no special improvement upon the other current definitions. It is a *suggestion* rather than a definition. Of late the statement that a straight line is determined by two of its points has gained favor as a definition, and it is true that a single particular straight line is by two points of it determined to be that several and singular straight line *after* it is known to be straight. But two points do not determine *the* straight line in general, that is to say, no two points of it being given will avail in the least as a test whereby to determine whether or not a line in question as to its straightness is really straight.

What is needed in a definition of the straight line is a statement or conspiracy of statements that shall express and exhibit to the intelligence the matters of fact in virtue of which it has that quality we call straightness. If this can be done in any other way than by defining a rule according to which the points that stud it are distributed so as to make it straight, then I am at a loss to conceive what that way can be. Moreover the straight line ought to be defined so as to be put right out in space in perfect self-sufficiency. I shall later on submit a definition that seems to me to fulfil the requisites I have mentioned, but for the present I must keep to the ways and results of Lobatchevsky.

Lobatchevsky begins his original matter with his Theorem 16 as follows:

> "All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *not-cutting*.
>
> "The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line.*
>
> "From the point A (Fig. 1) let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE.
>
> "In the right angle EAD either will all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines,

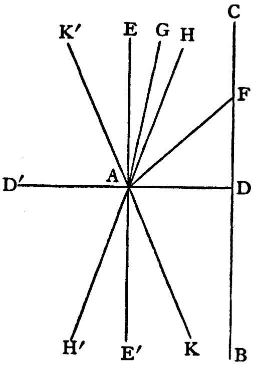


Fig. 1.

for example AG, which do not cut DC, how far soever they may be prolonged. In passing over from the cutting lines, as AF, to the not-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

"The angle HAD between the parallel HA and the perpendicular AD is called the parallel angle (angle of parallelism), which we will here designate by $\Pi(p)$ for $AD = p$.

"If $\Pi(p)$ is a right angle, so will the prolongation AE′ of the perpendicular AE likewise be parallel to the prolongation DB of the line DC, in addition to which we remark that in regard to the four right angles, which are made at the point A by the perpendiculars AE and AD, and their prolongations AE′ and AD′, every straight line which goes out from the point A, either itself or at least its prolongation, lies in one of the two right angles which are turned toward BC, so that except the parallel EE′ all others, if they are sufficiently produced both ways, must intersect the line BC.

"If $\Pi(p) < \frac{1}{2}\pi$, then upon the other side of AD, making the same angle $DAK = \Pi(p)$ will lie also a line AK, parallel to the prolongation DB of the line DC, so that under this assumption we must also make a distinction of *sides in parallelism.*

"All remaining lines or their prolongations within the two right angles turned toward BC pertain to those that intersect, if they lie within the angle $HAK = 2\Pi(p)$ between the parallels; they pertain on the other hand to the non-intersecting AG, if they lie upon the other sides of the parallels AH and AK, in the opening of the two angles $EAH = \frac{1}{2}\pi - \Pi(p)$, $E'AK = \frac{1}{2}\pi - \Pi(p)$, between the parallels and EE′ the perpendicular to AD. Upon the other side of the perpendicular EE′ will in like manner the prolongations AH′ and AK′ of the parallels AH and AK likewise be parallel to BC; the remaining lines pertain, if in the angle K′AH′, to the intersecting, but if in the angles K′AE, H′AE′ to the non-intersecting.

"In accordance with this, for the assumption $\Pi(p) = \frac{1}{2}\pi$ the lines can be only intersecting or parallel; but if we assume that $\Pi(p) < \frac{1}{2}\pi$, then we must allow two parallels, one on the one and one on the other side; in addition we must distinguish the remaining lines into non-intersecting and intersecting.

"For both assumptions it serves as the mark of parallelism that the line becomes intersecting for the smallest deviation toward the side where lies the parallel, so that if AH is parallel to DC, every line AF cuts DC, how small soever the angle HAF may be."

This long quotation is unavoidable unless one would risk a charge of misrepresentation or garbling. It states the full substance of the author's peculiar initial premises, those that distinguish his geometry from the geometry of Euclid and his disciples. With these premises and with about a dozen theorems adopted from the ordinary Euclidean geometry he develops a sequence of theorems as follows:

"17. A straight line maintains the characteristic of parallelism at all its points."

"18. Two lines are always mutually parallel."

"19. In a rectilineal triangle the sum of the three angles can not be greater than two right angles."

"20. If in any rectilineal triangle the sum of the three angles is equal to two right angles, so is this also the case for every other triangle."

"21. From a given point we can always draw a straight line that shall make with a given straight line an angle as small as we choose."

"22. If two perpendiculars to the same straight line are parallel to each other, then the sum of the three angles in a rectilineal triangle is equal to two right angles."

In the course of this Theorem 22 he goes on to remark:

"It follows that in all rectilineal triangles the sum of the three angles is either π and at the same time also the parallel $\Pi(p) = \frac{1}{2}\pi$ for every line p, or for all triangles this sum is $< \pi$ and at the same time also $\Pi(p) < \frac{1}{2}\pi$.

"The first assumption serves as *foundation for the ordinary geometry and plane trigonometry.*

"The second assumption can likewise be admitted without leading to any contradiction in the results, and founds a new geometric science,....which I intend here to expound as far as the development of the equations between the sides and angles of the rectilineal and spherical triangle."

Then follow Theorems 23, 24 and 25:

"23. For every given angle α we can find a line p such that $\Pi(p) = \alpha$."

"24. The farther parallel lines are prolonged on the side of their parallelism, the more they approach one another."

"25. Two straight lines which are parallel to a third are also parallel to one another."

With the completion of Theorem 25 the basis of the system of Lobatchevsky is fully laid. Theorem 23 is very inconspicuous in its enunciation but it contains matters of high significance as we shall see later on.

Now it is *true* that the assumption that the angle-sum of any triangle is less than two right angles leads to no contradictory results. If it is true for all triangles, it, of course, must be true for the isosceles right-angled triangle.

The existence of right angles and of right-angled triangles is pervadingly taken for granted by Lobatchevsky.

Now a right-angled isosceles triangle may be dissected into two other half size right-angled isosceles triangles by a line drawn from the mid-point of the hypotenuse to the vertex of the right angle, and then the two secondary triangles may in precisely like manner be each dissected into two tertiary right-angled isosceles triangles. The proof that the two secondary triangles are exactly equal to one another, that they are right-angled and isosceles, and that the four tertiary triangles are in all respects precisely in the same case is so simple in more than one way, that it would be almost an imputation upon the reader to spread it before him. But the right angle of the original triangle is a right angle, neither more nor less so that any deficiency of the angle-sum from two right angles must reside, if anywhere, in the two acute angles, and these being equal to one another, each must bear half of that deficiency. But any one of said acute angles of the original triangle is exactly equal to that acute angle of the same secondary triangle in which it belongs, that has its vertex at the vertex of the right angle of the original triangle, and this angle is precisely half the said last-mentioned (right) angle.

True, we may conceive that the bisection of an angle in some way operates to bend sharply the plane of the figure along the line of the bisecting line so that the lines that bound the right angle shall approach each other, but then the angle would no longer be a right angle, and the plane of the original triangle would be that rather curious form, a bent plane.

This leads me to remark somewhat out of order that Euclid's axioms 2 to 5 inclusive are not true in general. They may be false as applied to the addition and subtraction of angles unless the angles lie in the same plane.

But bending along section lines is not at all the kind of alteration that Lobatchevsky and his disciples admit. They will not for a moment concede that their straight lines are only approximately straight, and though through all of their illustrations they borrow the use of curved lines and of surfaces that have no straight lines, they yet insist that the lines they refer to as straight are in all sincerity completely and rigorously straight. Since we have no applicable criterion of straightness we can see how hard it is to invent any crucial test.

This claim that the assumptions of Lobatchevsky lead to no contradictory results has been so conspicuously proclaimed and moreover it is so plainly the very core of all the import of the new geometry that it sets a person of a skeptical turn of mind to wondering when and where and how any adequate tests of his assumptions have been made. Not certainly in Lobatchevsky's little brochure. That contains in all only thirty-six separate theorems and of these all but twelve are in perfect harmony with Euclidean assumptions. Even of these twelve three are nothing but rather elaborate definitions leaving only nine theorems on which to rest the claim of no contradictory results. And even as to these nine we shall, I think, later on see something pertinent to the weight of their argument. It is a little curious too to observe that of the non-Euclideans only Lobatchevsky and Bolyai deal with the matters in question by synthetical methods; all the rest are analysts.

But let us now look a little at the "angle of parallelism" with its other face, the deficiency of the angle-sum from two right angles, and see what it is and to what it leads. It must never be forgotten that the angle of parallelism is not proposed as a constant angle but is said to depend for its extent upon the extent of the line which makes with the parallel the angle of parallelism. It is not Π simply and constantly but Π (p), the (p) standing to mark the dependence of the angle upon the extent of the line mentioned.

Had Lobatchevsky used the phraseology of the differential calculus he would have said that the angle of parallelism is a function of the line which makes with the parallel the said angle of parallelism, the said line standing as the independent variable. For every length of the line (p) there is supposed to be a different angle of parallelism, and since these lengths are infinite in number we of necessity have to do business with an infinite number of angles of parallelism. When the independent variable is in the close neighborhood of infinity the angle of parallelism is taken to be in the close neighborhood of zero and at the very limit the parallel is there taken to be coincident and for a moment, at least, current with the line that elsewhere makes with it the angle of parallelism (Theorem 23, paragraph 4).

In view of this and taking also into account various phrases current among the disciples of the modern Zeno. such as curvature of space, the divergence of perpendiculars, etc., the inexpert, and I surmise some of the non-inexpert, mathematicians would be apt to feel misgiving lest there may have become insinuated into the whole doctrine some subtile fallacy. With such volatile elements to work with, it would be no wonder if estimates were more or less loose and floating.

There is ground for suspicion that the countenance given to non-Euclidean geometry by a number of eminent mathematical experts has somewhat overawed others that are very meagrely satisfied. These experts have wrapped up the doctrine in what to many is a maze of analytical language that requires a good deal of analytical erudition to compass and thoroughly possess. It makes the logician inclined to ask if these analysts have not mistaken some mere grammatical collocation of their analytical language for a real ideal possibility. We can say "round square," but nevertheless a "round square" is an absurdity. May not such analysts have made similar constructions? We shall later on say something more on this point.

Lobatchevsky however delivered his doctrine synthetically, and it is with his version that our present note of inquiry is concerned.

There is another thing too about the "angle of parallelism" that challenges attention. He says of the lines radiating from A and intermediate between the (p) line and the perpendicular thereto at A that they divide into two classes, lines that cut DC and lines that do not cut DC. But he definitely puts his parallel among the lines that do not cut. But how about the relation of that parallel to the next line, that is, the line that is the last of the lines that cut DC? Does it make an angle with the parallel or is it the same line? Do we not here begin to touch the very heart of the problem to find it a plain case of Zenoism?

If the lines make an angle I suppose that that angle can be bisected, indeed n-sected, and such section-lines will be lines that neither cut nor non-cut. If the lines are only one single line then we have a line that both cuts and noncuts. In short, we have the ever-recurring puzzle of how to formulate continuity.

I have long had an opinion of my own as to the true avenue of reconciliation and without any sort of pretense that said opinion is especially precious, I here state it for whatever it may prove to be worth.

In dividing the lines in question into cutting and noncutting lines Lobatchevsky observes the logical law of contradiction, viz., Any A is not any not-A. This law has two applications, subjective and objective, and the objective application includes our ideas when the same are objectively regarded. In its subjective application the law is, at least in our present state of intellectual development, insuperable. Its observation seems to be an indispensable condition for thought at all. So too in its objective application to things naturally discrete from one another there is at least, no occasion to dispense with its universality and necessity. But in its objective application to matters that involve the continuity of the objects considered the law of contradiction is not always of necessary compulsion, but at the boundary where A and not-A merge with one another we have the right, and as I say the perfectly logical right to regard the boundary specializations either as A or as not-A, just as one or the other fiat of ours may be suitable for our turn.

There is another thing about Lobatchevsky's parallel that ought to be emphasized. His "angle of parallelism" he makes less than a right angle but he does not ask for any *finite* lessness. *Any* concession from a right angle even though it be infinitesimal, aye, even though it be an infinitesimal of the infinitieth order, will satisfy his demands. And at the other end of the parallel he only asks that it shall just clear the perpendicular, and *any* clearance even of the most infinitesimal sort will do. In fact this sort of clearance is just what he does expressly demand. Now an infinitesimal is just precisely that sort of a quantity that *no finite number thereof will avail to make up any finite quantity.* Yet in his Theorem 23, viz., "*For every given angle* α *we can* find a line p such that $\Pi(p) = \alpha$" he proceeds to state his construction thus:

> "Let AB and AC (Fig. 10) be two straight lines which at the intersection point A make the acute angle α; take at random on AB

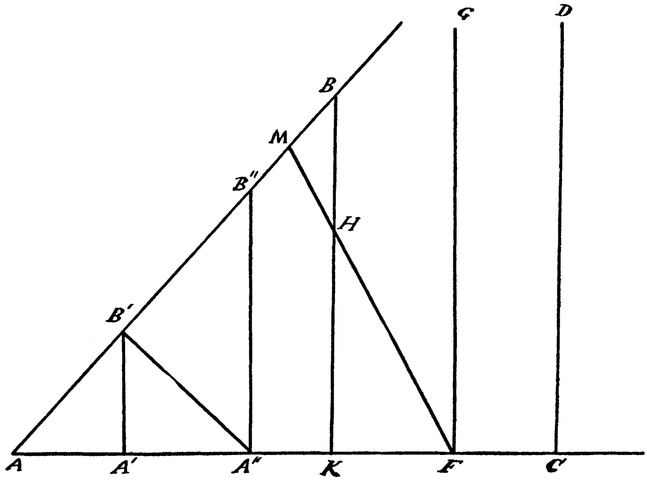


Fig. 10.

> a point B'; from this point drop B'A' at right angles to AC; make A'A'' = AA'; erect at A'' the perpendicular A''B''."

And then we find him going on as follows:

> "And so continue until a perpendicular CD is attained, which no longer intersects AB. This must of necessity happen, for if in the triangle AA'B' the sum of all three angles is equal to $\pi - \alpha$, then in the triangle AB'A'' it equals $\pi - 2\alpha$, in the triangle AA''B'' less than $\pi - 2\alpha$ (Theorem 20), and so forth, until it finally becomes negative and thereby shows the impossibility of constructing the triangle."

That "and so forth until" is richly monitory of the Zenonian "and so on *ad infinitum.*"

Now the validity of the construction proposed all depends upon whether a, the deficiency of the angle-sum from two right angles is finite or infinitesimal, and the fact that Lobatchevsky does not and will not expressly commit himself to *any* finite deficiency, is what his brochure as a whole makes abundantly manifest. Yet in spite of that we find him here in the very first theorem after he starts with his peculiar assumptions completed—a theorem in virtue of which alone his peculiar results emerge—covertly assuming that the deficiency is a finite deficiency.

The difficulty of proving the parallel postulate of Euclid resides, as I suppose, simply and solely by reason that it affirms the meeting of the lines proposed when they make interior angles on the same side are less than two right angles; that is to say: Euclid widens his postulate so as to be general and so as to include angle-sums that differ from two right angles by only an infinitesimal amount. Had he said "less than two right angles by a finite amount of angle" all the special and peculiar difficulty would, I suppose, disappear, for if the lines approach one nth of any distance from the intersection of the transversal, for that distance, then the lines would meet at n times that distance from the transversal. Of course the thus restricted postulate would not answer for geometrical purposes, and Euclid respected its debility.

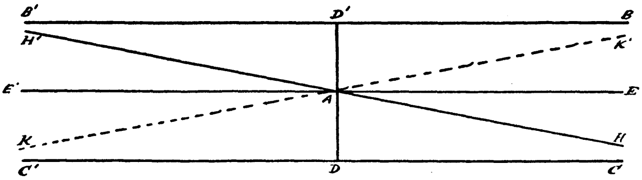
* * *

But it is time to search for results ourselves. It seems to me a pretty plain proposition that if the angle-sum of the triangle may be, and is supposed to be, less than two right angles, then such a figure as a plane rectangle is impossible. There is not angle stock enough in the two triangles into which a plane quadrilateral may be bisected to make up four right angles. A plane quadrilateral may on the assumption in question have one, two, perhaps three, but *not* four right angles. As a corollary a square is impossible and the cube is in like case.

Since the service of arithmetic, algebra, and all mathematical analysis in geometry is due entirely to the exact appropriation of those branches of science to the Euclidean assumptions, may it not well be wondered what good and true services they can render on behalf of assumptions that conflict with those of Euclid? When rectangles, squares, and cubes are impossible what geometrical meaning can products and quotients, powers and roots have? And how can addition and subtraction manage to straddle congruently between the Euclidean and the non-Euclidean products and powers, quotients and roots? How analysis can confirm non-Euclidean assumptions seems to me to be a matter needing explanation.

I will now proceed to state and explain certain considerations that seem to me to be wholly irreconcilable with any claim that the assumptions of Lobatchevsky lead to no contradictory results and that utterly prevent the same from presenting that harmony that is the mark of true science.

Consider the figure following.



AD is here the (p) line, AH the parallel making the angle of parallelism HAD. DC is the perpendicular to the (p) line at D, and AE the perpendicular to the same line at A. Draw also through A the line AK, making the angle KAD equal to the angle of parallelism HAD. Now prolong all of these lines, except the (p) line, boundlessly in straight lines but in the reverse sense: DC towards and beyond C′, AE towards and beyond E′, AH towards and beyond H′, and AK towards and beyond K′. Prolong the (p) line to D′, making AD′= AD and through D′ draw the boundless straight line BD′B′ perpendicular to DAD′. Then the angle H′AD′ will be equal to the angle of parallelism HAD, according to Theorem 6 (vertical angles). Both will be angles of parallelism Π (p). AD′ will be a replica of the (p) line and DD′=2(p). In short the figure H′AD′B′ will be in all respects the same case as is the case of the figure HADC. H′A will be parallel to B′D′ by the very same manner of token that HA is parallel to DC, and since "a straight line maintains the characteristic of parallelism at all its points" (Theorem 17) the whole line H′H is parallel to B′D′; and again, since "two lines are always mutually parallel" (Theorem 18) B′D′ is parallel to H′A and to H′AH, and still again by Theorem 17 B′D′B is parallel to H′AH. By precisely the same manner of token the line H′AH is shown to be parallel to C′DC and C′DC parallel to H′AH. But "two straight lines which are parallel to a third are also parallel to one another" (Theorem 25) so that B′B is parallel to C′C and both being perpendicular to DAD′ it seems to be shown that *some* lines at least that are each perpendicular to the same transversal are parallel to each other, and it also seems to be shown that whenever the (p) line is double some other instance of its species the angle of parallelism is a right angle.

This looks to me very much like a proof that in all cases the angle of parallelism is a right angle.

Now by the very same course of deduction (no step of which is unsanctioned in the "system" of Lobatchevsky) the line KAK′ is shown to be parallel to HAH′ and to EAE′, in spite of the rather important feature that they cut one another at A.

So, unless I am altogether mistaken in the above, we find ourselves in view of a very curious and remarkable phase in the history of geometric research. We find Lobatchevsky hitting upon the right and sufficient way of proving the parallel postulate of Euclid. We find him pursuing that way with eminent success for a while, but at Theorem 23 getting shunted by the confounding of the infinitesimal with the finite.

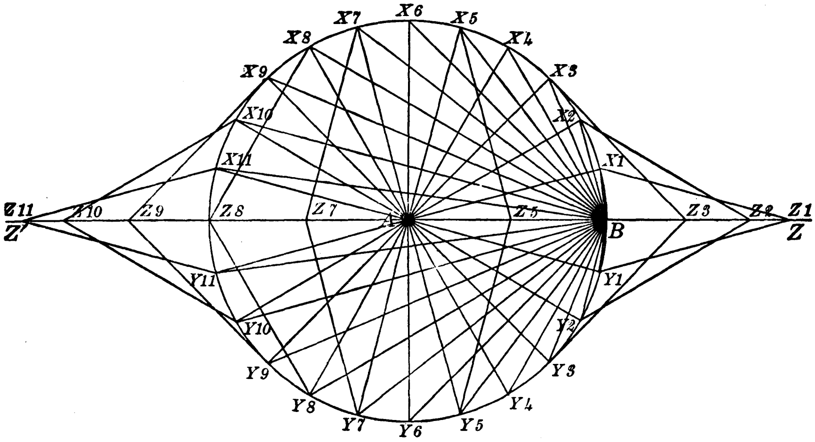
Still it may be that there is something about the matter that I do not understand. If so I can only protest that my failure is not due to any lack of very respectful (I do not want to say absurdly respectful) study of Lobatchevsky's little brochure.

Considering that the modern man, aye, even the modern man of so-called liberal education, finds himself lost between a disposition on his part to respect the august majesty of mathematics and a disposition tempting him to regard the non-Euclideans something as the mathematician regards the circle-squarer and the perpetual motionist, it would seem as though it might be worth the while for some one of the non-Euclidean sect to so explain their doctrine as to make it manifestly clear and sound to minds as unable as mine.

It is very much to be wished that mathematicians would have less contempt for the philosophers and that the philosophers would follow less that esteem for mathematical power of survey and penetration that led Sir William Hamilton to judge them only able to walk straight in a ditch dug by others. In Dr. Paul Carus we have an instance all too rare of a philosopher fully up to date and fully furnished in mathematics so far as the same has any philosophical import. His *Foundations of Mathematics* deals with the questions now most in gestation in a way and with a mastery that the mathematicians can ill afford to neglect. In particular the three cardinal conceptions of anyness, uniquity and even boundaries are therein, so far as I am aware, first put forward and exhibited in clear relief as the most significant and consequential ideas that contribute to the foundations of mathematics.

* * *

To my mind it is the calamity of geometry that it falls down at its very start in not providing a thoroughly competent definition of the straight line. The conception is pervadingly necessary everywhere in mathematics. All things in mathematics have been made by it, and without it has not been made anything that has been made; and all the modern questions concerning the foundations of geometry—the nature, origin and meaning of axioms and the like—are embarrassed to the point of insolubility simply and solely by reason of the lacking definition. I hope, therefore, I shall be pardoned on account of my good intentions, if for nothing else, when I offer for scrutiny and judgment a definition which so far as I know is a new one and seems to me to depend upon nothing experiential in its nature except the unalterability of the interval. I show, I think, how the ruler may be derived by means of the compass alone. As an introduction consider the figure following:



It is constructed thus: Take any two points, say A and

B. With, say, A as a turn-point (it might just as well have been B) and with the interval AB as the compass opening, scribe the circle Bx_1 x_2 etc. clear around complete. Then with B as turn-point and with *any* opening of the compass, short of 2AB, mark off on the first circle two points, say x_1 and y_1. The same will be, of course, at equal intervals from B. Then from each of the points so marked scribe circles with the compass opening the interval AB. Such circles will all pass through A, but besides that they will otherwhere intersect and determine a point as, say z_1. Now for each possible pair of points thus determined on the first circle, there will be thus determined by the circle intersections lastly above mentioned a point, and each and all of these last mentioned points *will lie in a straight line.*

But stop! I have been talking as though I assumed the plane to be already earned and known, and until we duly and geometrically earn the plane we have no geometric right to use it. I have employed my figure and my comment thereupon just to lead the imagination a little so that it will easier picture and understand the real figure I desire to use, which cannot be drawn upon paper.

I now say, Take any two separate points, say A and B. With either of them, say A, as a turn-point and with the interval AB scribe a *sphere.* Then from B as a turn-point and with any interval short of 2AB scribe any secondary *sphere.* The infinitely numerous thus possible secondary spheres will each intersect the primary sphere in a curve (in fact a circle). On any such curve select at random any triad of separate points, and with each so selected point as a turn-point and with the interval AB as radius scribe tertiary spheres, three in all. Such tertiary spheres will all pass through A, but they will besides at another place intersect in and determine a point. Now for each of the infinitely numerous point triads prepared for and selected, as I have prescribed, there will be thus determined a single and *unique* point, and all these so determined and last mentioned points will lie in a straight line. The limited straight line so formed or rather the infinitely numerous array of points thus determinable and that stud the line, I call The Straight Range, and it is easy to see that it will be 4AB in extent.

I define it thus: The Straight Range is a continuous linear array of points such that any point of it is separated from each one of some triad of separate points by a common interval X and each one of said triad of separate points is at once at the interval X from one of a pair of points that are separated by the interval X and at a common interval Y from the other one of said pair of points.

No doubt a draftsman would have to be eminently expert to locate some of the points with precision, but that does not much detract from the scientific value of the construction.

Just as we center the primary sphere at A so we may also center the same at B and proceed to construct a range in all respects similarly as when A is taken. The two ranges will perfectly coincide as to three-fourths of their several and respective extents and so all but the extreme ends of the combined range 5AB will be perfectly determined even for the draftsman.

The chief use of the straight range construction is the insight or atsight it gives us as to the nature of the straight line. It enables us to see just what it is that is the essence of the straightness of the straight line, viz., the perfect uniquity of any point of any triad of points upon it in respect to the joint pair of intervals that separate it from the other points of said triad of points. It is the *only point that exists* that has at once and jointly those intervals. So I define the straight line as follows:

I first define *The Straight Point-Triad* thus:

The Straight Point-Triad is a triad of points such that any point of the triad is the only point that exists that has together at once the same pair of intervals that it has from the other points of the triad respectively. Then:

The Straight Line is a line such that any triad of points upon it is a straight point-triad.

Suppose we now venture to define the plane by its points. Conformably with the above definition I would say that it takes four points to define a plane, and I would first define the plane point-tetrad thus:

The Plane Point-Tetrad is a point-tetrad such that no point-triad of it being a straight point-triad any point of it is the only point that exists that together at once has the same triad of intervals that it has from the other points of the point-tetrad respectively.—Then:

The Plane is a surface such that any point-tetrad upon it that has no straight point-triad is a plane point-tetrad.

Plane point-tetrads divide into two sorts according to whether one of the points is within the plane opening between the other three points, or whether no one of the points is within said opening. Call the former sort *close* plane point-tetrads and the latter sort *convex* plane point-tetrads. In every close plane point-tetrad say, $a\ b\ c\ d$, where, say d is the point within the others, there exists for each of the latter a point x_n such that $a\ d\ x_1$, $b\ d\ x_2$ and $c\ d\ x_3$ will all of them be straight point-triads. In every convex plane point-tetrad the other points stand with relation to any one of them as adjacent points two in number or as a single opposite point, and a single point y *exists* such that it forms with the adjacent points a straight point-triad and with the opposite points another straight point-triad. But I have not yet put together enough of "The Elements of Compass Geometry" to make it worth while to pursue here whither it will lead. But I hope I am not mistaken in my faith that geometry depends at last upon one non-subjective datum alone, to wit: the invariance of the interval, and I furthermore avow my faith that this datum is indispensable, ineluctable.

Prof. Cayley showed that everything in non-Euclidean geometry could be perfectly presented in ordinary Euclidean space (as it is called) by suitably varying the notion of distance. But the necessity of the fixed unalterable interval as a foundation is not thus to be surmounted, for variation itself has no meaning, norm or description when it lacks the basis and background of the invariable interval.

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